Problem 1. Given triangle $ABC$ the point $J$ is the centre of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$.

Prove that $M$ is the midpoint of $ST$.

(The excircle of $ABC$ opposite the vertex $A$ is the circle that is tangent to the line segment $BC$, to the ray $AB$ beyond $B$, and to the ray $AC$ beyond $C$.)

Problem 2. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2a_3 \cdots a_n = 1$. Prove that
\[ (1 + a_2)^2(1 + a_3)^3 \cdots (1 + a_n)^n > n^n. \]

Problem 3. The liar’s guessing game is a game played between two players $A$ and $B$. The rules of the game depend on two positive integers $k$ and $n$ which are known to both players.

At the start of the game $A$ chooses integers $x$ and $N$ with $1 \leq x \leq N$. Player $A$ keeps $x$ secret, and truthfully tells $N$ to player $B$. Player $A$ now tries to obtain information about $x$ by asking player $A$ questions as follows: each question consists of $B$ specifying an arbitrary set $S$ of positive integers (possibly one specified in some previous question), and asking $A$ whether $x$ belongs to $S$. Player $B$ may ask as many such questions as he wishes. After each question, player $A$ must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k + 1$ consecutive answers, at least one answer must be truthful.

After $B$ has asked as many questions as he wants, he must specify a set $X$ of at most $n$ positive integers. If $x$ belongs to $X$, then $B$ wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then $B$ can guarantee a win.
2. For all sufficiently large $k$, there exists an integer $n \geq 1.99^k$ such that $B$ cannot guarantee a win.
Wednesday, July 11, 2012

**Problem 4.** Find all functions \( f : \mathbb{Z} \to \mathbb{Z} \) such that, for all integers \( a, b, c \) that satisfy \( a + b + c = 0 \), the following equality holds:

\[
f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).
\]

(Here \( \mathbb{Z} \) denotes the set of integers.)

**Problem 5.** Let \( ABC \) be a triangle with \( \angle BCA = 90^\circ \), and let \( D \) be the foot of the altitude from \( C \). Let \( X \) be a point in the interior of the segment \( CD \). Let \( K \) be the point on the segment \( AX \) such that \( BK = BC \). Similarly, let \( L \) be the point on the segment \( BX \) such that \( AL = AC \). Let \( M \) be the point of intersection of \( AL \) and \( BK \).

Show that \( MK = ML \).

**Problem 6.** Find all positive integers \( n \) for which there exist non-negative integers \( a_1, a_2, \ldots, a_n \) such that

\[
\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.
\]