Problem 1. An acute-angled triangle \(ABC\) has orthocentre \(H\). The circle passing through \(H\) with centre the midpoint of \(BC\) intersects the line \(BC\) at \(A_1\) and \(A_2\). Similarly, the circle passing through \(H\) with centre the midpoint of \(CA\) intersects the line \(CA\) at \(B_1\) and \(B_2\), and the circle passing through \(H\) with centre the midpoint of \(AB\) intersects the line \(AB\) at \(C_1\) and \(C_2\). Show that \(A_1, A_2, B_1, B_2, C_1, C_2\) lie on a circle.

Problem 2. (a) Prove that
\[
\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1
\]
for all real numbers \(x, y, z\), each different from 1, and satisfying \(xyz = 1\).

(b) Prove that equality holds above for infinitely many triples of rational numbers \(x, y, z\), each different from 1, and satisfying \(xyz = 1\).

Problem 3. Prove that there exist infinitely many positive integers \(n\) such that \(n^2 + 1\) has a prime divisor which is greater than \(2n + \sqrt{2n}\).
Thursday, July 17, 2008

Problem 4.  Find all functions $f : (0, \infty) \to (0, \infty)$ (so, $f$ is a function from the positive real numbers to the positive real numbers) such that
\[
\frac{(f(w))^2 + (f(x))^2}{f(y)^2 + f(z)^2} = \frac{w^2 + x^2}{y^2 + z^2}
\]
for all positive real numbers $w, x, y, z$, satisfying $wx = yz$.

Problem 5.  Let $n$ and $k$ be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let $N$ be the number of such sequences consisting of $k$ steps and resulting in the state where lamps $1$ through $n$ are all on, and lamps $n + 1$ through $2n$ are all off.

Let $M$ be the number of such sequences consisting of $k$ steps, resulting in the state where lamps $1$ through $n$ are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine the ratio $N/M$.

Problem 6.  Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles $ABC$ and $ADC$ by $\omega_1$ and $\omega_2$ respectively. Suppose that there exists a circle $\omega$ tangent to the ray $BA$ beyond $A$ and to the ray $BC$ beyond $C$, which is also tangent to the lines $AD$ and $CD$. Prove that the common external tangents of $\omega_1$ and $\omega_2$ intersect on $\omega$. 

Language: English  Time: 4 hours and 30 minutes
Each problem is worth 7 points